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Assignment I: MTH 512

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* Question 1.

i) The smallest n so that $Q_1, Q_2, Q_3 \in \mathbb{R}^n$ is 4

since if we take $n=3$ and $Q_1 = (1,0,0)$, $Q_2 = (0,1,0)$, $Q_3 = (0,0,1)$ $\in \mathbb{R}^3$

then $\text{span}\{Q_1, Q_2, Q_3\} = \mathbb{R}^3$ (contradiction)

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ii) To prove that Q_1+Q_2, Q_1+Q_3, Q_2+Q_3 are independent in $\mathbb{R}^n$ we need to prove that the only solution of:

$a(Q_1+Q_2) + b(Q_1+Q_3) + c(Q_2+Q_3) = 0$ is $a=b=c=0$

so $aQ_1 + aQ_2 + bQ_1 + bQ_3 + cQ_2 + cQ_3 = 0$

$\Rightarrow (a+b)Q_1 + (a+c)Q_2 + (b+c)Q_3 = 0$

since Q_1, Q_2 and Q_3 are independent in $\mathbb{R}^n$ then

$a+b=0$, $a+c=0$, and $b+c=0$

$\Rightarrow \left.\begin{matrix} b=-a \\ c=-a \end{matrix}\right\}$ $-a-a=0 \Rightarrow -2a=0 \Rightarrow a=b=c=0$

thus (Q_1+Q_2), (Q_1+Q_3), (Q_2+Q_3) are independent in $\mathbb{R}^n$

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iii) we know that Q_1, Q_2 and Q_3 are orthogonal

and $Q = a_1 Q_1 + a_2 Q_2 + a_3 Q_3$

. for a_1 : $Q.Q_1 = a_1 Q_1 Q_1 + a_2 Q_2 Q_1 + a_3 Q_3 Q_1$

$\Rightarrow Q.Q_1 = a_1 \|Q_1\|^2 + 0 + 0$ (because they are orthogonal)

$\Rightarrow a_1 = \dfrac{Q.Q_1}{\|Q_1\|^2}$

. for a_2 : $Q.Q_2 = a_1 Q_1 Q_2 + a_2 Q_2 Q_2 + a_3 Q_3 Q_2$

$Q.Q_2 = 0 + a_2 \|Q_2\|^2 + 0$

$\Rightarrow a_2 = \dfrac{Q.Q_2}{\|Q_2\|^2}$

. for a_3 : $Q.Q_3 = a_1 Q_1 Q_3 + a_2 Q_2 Q_3 + a_3 Q_3 Q_3$

$Q.Q_3 = 0 + 0 + a_3 \|Q_3\|^2$

$\Rightarrow a_3 = \dfrac{Q.Q_3}{\|Q_3\|^2}$

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* Question 2: $D = \text{span}\{(2a+3, -b+1, 6a-2b+11, 0) \mid a,b \in \mathbb{R}\}$

i) $D = \{(2a+3, -b+1, 6a+9-2b+2, 0) \mid a,b \in \mathbb{R}\}$

$= \{(2a+3)(1,0,3,0) + (-b+1)(0,1,2,0) \mid a,b \in \mathbb{R}\}$

$= \text{span}\{(1,0,3,0), (0,1,2,0)\}$

thus D is a subspace of $\mathbb{R}^4$ because it can be written as span.

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ii) let $Q_1 = (1,0,3,0)$ and $Q_2 = (0,1,2,0)$

$\{w_1, w_2\}$ is the orthogonal basis of D where:

- $w_1 = Q_1 = (1,0,3,0)$
- $w_2 = Q_2 - \dfrac{Q_2 \cdot w_1}{\|w_1\|^2} w_1$

$$= (0,1,2,0) - \frac{6}{10}(1,0,3,0)$$

$$= (0,1,2,0) - \left(\frac{3}{5}, 0, \frac{9}{5}, 0\right)$$

$$= \left(-\frac{3}{5}, 1, \frac{1}{5}, 0\right)$$

thus $\{(1,0,3,0), (-\frac{3}{5}, 1, \frac{1}{5}, 0)\}$ is the orthogonal basis of D.

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* Question 3: $A = \begin{bmatrix} 5 & 3 & 1 & 1 \\ 1 & 3 & -1 & 0 \\ 2 & -6 & 4 & 1 \\ 4 & -12 & -4 & 1 \end{bmatrix}$

To see if 6 is an eigenvalue of A, we need to show that $\{(0,0,0,0)\}$ is not the solution set of: $(6I_4 - A)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & -3 & -1 & -1 \\ -1 & 3 & 1 & 0 \\ -2 & 6 & 2 & -1 \\ -4 & 12 & 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left(\begin{array}{cccc|c} 1 & -3 & -1 & -1 & 0 \\ -1 & 3 & 1 & 0 & 0 \\ -2 & 6 & 2 & -1 & 0 \\ -4 & 12 & 4 & 5 & 0 \end{array}\right) \end{array} \xrightarrow[\begin{subarray}{c} 2R_1 + R_3 \to R_3 \\ 4R_1 + R_4 \to R_4 \end{subarray}]{R_1 + R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & -3 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x_4 = 0 \\ x_1 - 3x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_4 = 0 \\ x_1 = 3x_2 + x_3 \end{cases} \xrightarrow[\text{and } x_3 = b]{\text{take } x_2 = a} \begin{cases} x_1 = 3a + b \\ x_2 = a \\ x_3 = b \\ x_4 = 0 \end{cases}$$

so the solution set $= \{(3a+b), a, b, 0\} \mid a, b \in \mathbb{R}\}$

$= \cancel{\text{span}}\ \{a(3,1,0,0) + b(1,0,1,0) \mid a, b \in \mathbb{R}\}$

so $E_6 = \text{Span}\{(3,1,0,0), (1,0,1,0)\}$

thus 6 is an eigenvalue of A.

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. Question 3:

let $Q_1 = (3,1,0,0)$ and $Q_2 = (1,0,1,0)$

$\{w_1, w_2\}$ is the orthogonal basis for E_6 where

. $w_1 = Q_1 = (3,1,0,0)$

. $w_2 = Q_2 - \frac{Q_2 \cdot w_1}{\|w_1\|^2} w_1$

$= (1,0,1,0) - \frac{3}{10}(3,1,0,0)$

$= \left(\frac{1}{10}, -\frac{3}{10}, 1, 0\right)$

thus $\left\{(3,1,0,0), \left(\frac{1}{10}, -\frac{3}{10}, 1, 0\right)\right\}$ is the orthogonal basis for E_6.

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* Question 4: Let A be $n \times n$ matrix and r a fixed real number where ~~all~~ the sum of all numbers of each row of A is equal to r.

Now consider the non-zero point $Q = (1, 1, \dots, 1) \in \mathbb{R}^n$ (all entries of Q is 1)

$$. \; A Q^T = \underset{(n\times n)}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}} \underset{(n\times 1)}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}} = \begin{pmatrix} a_{11} + a_{12} + \cdots + a_{1n} \\ a_{21} + a_{22} + \cdots + a_{2n} \\ \vdots \\ a_{n1} + a_{n2} + \cdots + a_{nn} \end{pmatrix} = \underset{(n\times 1)}{\begin{pmatrix} r \\ r \\ \vdots \\ r \end{pmatrix}}$$

and $$r Q^T = r \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} r \\ r \\ \vdots \\ r \end{pmatrix}$$

thus $A Q^T = r Q^T$

So r is an eigen value of A because there exists a non-zero point in $\mathbb{R}^n$, Q, such that $A Q^T = r Q^T$

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* Question 5: $F = \begin{bmatrix} 0 & 0 & 4 & 6 \\ 1 & 3 & -1 & 0 \\ 0 & -6 & 4 & 1 \\ 4 & 12 & -4 & 2 \end{bmatrix}$

$$|F| = (-1)^{1+3} \cdot 4 \begin{vmatrix} 1 & 3 & 0 \\ 0 & -6 & 1 \\ 4 & 12 & 2 \end{vmatrix} + (-1)^{1+4} \cdot 6 \begin{vmatrix} 1 & 3 & -1 \\ 0 & -6 & 4 \\ 4 & 12 & -4 \end{vmatrix}$$

$$= 4 \left[(-1)^{1+1} \cdot 1 \begin{vmatrix} -6 & 1 \\ 12 & 2 \end{vmatrix} + (-1)^{1+2} \cdot 3 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} \right] - 6 \left[(-1)^{1+1} \cdot 1 \begin{vmatrix} -6 & 4 \\ 12 & -4 \end{vmatrix} + (-1)^{3+1} \cdot 4 \begin{vmatrix} 3 & -1 \\ -6 & 4 \end{vmatrix} \right]$$

$$= 4 \left[(-12-12) - 3(-4) \right] - 6 \left[(24-48) + 4(12-6) \right]$$

$$= 4(-24+12) - 6(-24+24)$$

$$= -48.$$

so $|F| = -48$

$D \xrightarrow{-2R_2} F$ thus $|F| = -2|D| \Rightarrow |D| = -\frac{1}{2}|F| = 24$

$C \xrightarrow{R_3 \leftrightarrow R_2} D$ thus $|D| = -|C| \Rightarrow |C| = -24$

$B \xrightarrow{-6R_1 + R_4 \to R_4} C$ thus $|C| = |B| \Rightarrow |B| = -24$

$A \xrightarrow{3R_2} B$ thus $|B| = 3|A| \Rightarrow |A| = \frac{1}{3}|B| = -8$

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* Question 6 :

i) $L = \{(a, b^3, 0) \mid a, b \in \mathbb{R}\}$

NO

$\in \mathbb{R}$

. axiom 1: Let $Q_1, Q_2 \in L$ then $Q_1 = \alpha_1 a + \alpha_2 b^3 + 0$ for some const α_1, α_2

$Q_2 = \beta_1 a + \beta_2 b^3 + 0$ for some const. $\beta_1 \beta_2$

$\in \mathbb{R}$ not a point

thus $Q_1 + Q_2 = (\alpha_1 + \beta_1) a + (\alpha_2 + \beta_2) b^3 + 0$

so $Q_1 + Q_2 \in L$

. axiom 2: Let $Q \in L$ and α be a constant

then $Q = \alpha_1 a + \alpha_2 b^3 + 0$

and $\alpha Q = (\alpha \alpha_1) a + (\alpha \alpha_2) b^3 + 0$

so $\alpha Q \in L$

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. axiom 3: take $a = b = 0$ thus $(0, 0, 0) \in L$

So L is a subspace of $\mathbb{R}^3$.

(ii) $L = \{(a, 0, b^2) \mid a, b \in \mathbb{R}\}$

Take $Q = (0, 0, 4) \in L$ where $a = 0$ and $b = 2$

and consider $\alpha = -1$

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thus $\alpha Q = (0, 0, -4) \notin L$

thus L is not a subspace of $\mathbb{R}^3$ since axiom 2 fails.

* Question 6:

iii) $L = \{ (b, b^3, 0) \mid b \in \mathbb{R} \}$

Take $Q = (1, 1, 0) \in L$ such that $b = 1$

and consider $\alpha = 2$

then $\alpha Q = (2, 2, 0) \notin L$

thus L is not a subspace of $\mathbb{R}^3$

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iv) Let us find the solution set of $(3 I_4 - A) X = 0$

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 \quad \cancel{\frac{1}{3}R_2}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right)$$

$$\downarrow R_1 + R_2 \to R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right) \xleftarrow{\frac{1}{3}R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right)$$

$$\downarrow R_2 + R_3 \to R_3$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4/3 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{R_3+R_4\to R_4}\left(\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&-4/3&0\\0&0&0&5/3&0\end{array}\right)$$

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$$\xrightarrow{\frac{3}{5}R_4}\left(\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&-4/3&0\\0&0&0&1&0\end{array}\right)\xrightarrow{\frac{4}{3}R_4+R_3\to R_3}\left(\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{array}\right)$$

thus Solution set of $(3I_4-A)X=0$ is $\{(0,0,0,0)\}$

therefore 3 is not an eigenvalue of A.

v) let A be a 4x4 matrix where $A_2=A_4$ (columns)

solve the system $A\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=A_2$

$$\Rightarrow A_2 = x_1A_1+x_2A_2+x_3A_3+x_4A_4$$

$$A_2 = x_1A_1+(x_2+x_4)A_2+x_3A_3$$

thus $\begin{cases}x_1=0\\x_2+x_4=1\\x_3=0\end{cases}\Rightarrow\begin{cases}x_1=0\\x_2=1-x_4\\x_3=0\end{cases}\xrightarrow{\text{take } x_4=a}\begin{cases}x_1=0\\x_2=1-a\\x_3=0\\x_4=a\end{cases}$

thus the solution set of $AX=A_2=\{(0,1-a,0,a)\mid a\in\mathbb{R}\}$

so it has infinitely many solutions

and $\underset{(a=1)}{(0,0,0,1)}$, $\underset{(a=2)}{(0,-1,0,2)}$, $\underset{(a=3)}{(0,-2,0,3)}$ are 3 distinct points that belong to the solution set of the system.